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Symmetries of linear ordinary differential equations

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Abstract. We discuss the Lie symmetry approach to homogeneous, linear, ordinary differential equations in an attempt to connect it with the algebraic theory of such equations. In particular, we pay attention to the fields of functions over which the symmetry vector fields are defined and, by defining a noncharacteristic Lie subalgebra of the symmetry algebra, are able to establish a general description of all continuous symmetries. We use this description to rederive a classical result on differential extensions for second-order equations.

1. Introduction

As a practical tool in the analysis of differential equations, particularly in the construction of exact solutions, Lie symmetry algebras and groups play a prominent role. Thus, the theory of special functions can be regarded as the representation theory of finite-dimensional symmetry groups of the Laplace equation and other, integrable cases of Hamiltonian systems on both finite- and infinite-dimensional spaces arise from more specialized, Hamiltonian group actions, and there are countless applications to specific physical problems (see, for example, [9, 10, 12]).

Lie point symmetries are continuous, one parameter Lie pseudogroups acting on the space of dependent and independent variables, extended to the space of all derivatives, which fix the variety defining the differential equation. They are generated by vector fields which for a given equation, comprise a Lie algebra. Solvability of the associated Lie group implies 'solvability' of the differential equation. The theory is applicable, in principle, to any analytic class of differential equation.

One or two issues are not addressed in the classical theory outlined above. The first is that there may be useful continuous, symmetries which are not generated by point transformations. Such generalized, 'dynamical' or 'hidden' symmetries [13, 5, 11, 1, 3] may not preserve the particular choice of derived variables. They are the fully geometric symmetries of the equation.

A second issue is that of the function spaces to which the coefficients in the vector fields belong and over which they form a Lie algebra. Clearly the size and dimensionality of the symmetry algebra depend on these. Consider in this context the statement often quoted from Lie [7] that the point symmetries of a second-order equation form an algebra of maximal dimension eight and that, if this bound is achieved, the equation is linearizable. It is possible that by restricting the coefficients of the symmetries to given differential fields, one may find a rich structure within the symmetry algebra and be able to deduce, for instance, the function class of the linearizing transformation. We will see in this paper that, for linear equations, the symmetry algebra viewed in this way sheds light on the details of the solution space.

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In this paper we use an abstract definition of symmetry framed in terms of exterior differential ideals over specified differential fields. This definition is too broad because it includes, within the symmetry algebra, an infinite-dimensional characteristic subalgebra of generalized symmetries which are, in practical terms, not useful. We are able to factorize this out to obtain the noncharacteristic symmetry algebra, S_Θ, for the ideal Θ. This algebra is, in general, infinite-dimensional over a specified field of constants but is also a vector space over the field of invariants of the characteristic subalgebra over which it may be finite-dimensional. We give the usual type of results about reducibility in terms of certain ideals contained in Θ.

For a large enough extension of a differential field we then present a complete description of S_Θ for the general linear, homogeneous equation of degree n. This is an infinite-dimensional algebra, over the field of constants, which is a finite-dimensional vector space over the invariants. The description is both natural and useful as we show in going on to apply it to the general second-order equation where we rederive a result from differential algebra.

We should emphasize that the noncharacteristic symmetry algebra is not the point symmetry algebra. Although it does contain a homomorphic image of the latter, it is both larger and has a much simpler structure. Part of our motivation in studying it is to understand the algebraic theory of differential field extensions using Lie symmetry.

For an introduction to the theory and some results of exterior differential algebra the reader might consult [2].

2. Definitions and fundamental results

Let k_0 be the (algebraically closed) field of constants of the zero characteristic differential field k with single derivation denoted by ∂_x. We denote extensions involving only this derivation by k_1, k_2, etc. We denote by $K = k[y_0, y_1, \ldots, y_{n-1}]$ the differential polynomial ring in the variables $y_0, y_1, \ldots, y_{n-1}$ with commuting derivations $\partial_x, \partial_0, \partial_1, \ldots, \partial_{n-1}$ satisfying $\partial_i(y_j) = \delta_{ij}$. Likewise, $K_1 = k_1[y_0, y_1, \ldots, y_{n-1}]$ etc. The derivations on these rings extend to their fields of fractions $\mathfrak{K} = k(y_0, y_1, \ldots, y_{n-1})$, $\mathfrak{K}_1 = k_1(y_0, y_1, \ldots, y_{n-1})$, etc.

Let $\bigwedge$ be the exterior differential algebra of differential forms of nonzero weight over $\mathfrak{K}$. It is generated as a (finite-dimensional) $\mathfrak{K}$-vector space by one-forms $\mathrm{d}x, \mathrm{d}y_0, \mathrm{d}y_1, \ldots, \mathrm{d}y_{n-1}$. T will be the corresponding $\mathfrak{K}$-vector space of vector fields, that is, derivations of $\mathfrak{K}$. Note that $\bigwedge$ is a $\mathfrak{K}$-algebra under exterior multiplication whereas T is only a k_0-algebra under the Lie bracket of vector fields. When we wish to talk about forms or vector fields with coefficients in one of the field extensions we will write, for example, $\mathfrak{K}_1 \bigwedge$ and $\mathfrak{K}_1 T$.

Let Θ be a differential ideal in $\bigwedge$. That is, in addition to being a ring ideal we have $\mathrm{d}\Theta \subseteq \Theta$. We write $A \trianglelefteq B (A \triangleleft B)$ when A is a (proper) differential ideal of B. We use $\leqslant$ ($<$) similarly for (proper) vector subspaces. For the purposes of this paper we assume that Θ is finitely generated by one-forms $\theta_1, \theta_2, \ldots, \theta_r$, linearly independent over $\mathfrak{K}$ in an open subset of $\mathbb{C}^{n+1}$, which is certainly the case for ideals associated with ordinary differential systems away from singular points. r is the *rank* of Θ. Thus, there exist elements (one-forms) $\Gamma_i^j \in \bigwedge$ such that,

$$\mathrm{d}\,\theta_i = \sum_{j=1}^{r} \Gamma_i^j \wedge \theta_j \qquad i = 1, \ldots, r. \tag{1}$$

This is the Frobenius integrability condition.

With the differential ideal Θ we associate two k_0-subalgebras of T, the Lie symmetry algebra, L_Θ, and the characteristic algebra, $\mathcal{X}_\Theta$. $\mathcal{L}_X$ is the Lie derivative operator.

Definition 1. The *Lie symmetry algebra* of Θ in T is

$$L_\Theta = \{X \in T | \mathcal{L}_X \Theta \subseteq \Theta\}.$$

Definition 2. The *characteristic algebra* of Θ in T is

$$\mathcal{X}_\Theta = \{X \in T | X \lfloor \Theta \subset \Theta\}.$$

It is evident that these are indeed k_0 algebras from the relations,

$$\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y] \tag{2}$$

$$[X, Y] \lfloor \theta = X(Y \lfloor \theta) - Y(X \lfloor \theta) - \mathrm{d}\theta(X, Y) \tag{3}$$

θ being any one-form in Θ and the closure condition (1). If we denote by $\mathfrak{I}_\Theta$ the set $\{f \in \mathfrak{K} | X(f) = 0, \forall X \in \mathcal{X}_\Theta\}$, then L_Θ is a k_0-algebra but an $\mathfrak{I}_\Theta$-vector space. Likewise, $\mathcal{X}_\Theta$ is a $\mathfrak{K}$ vector space and also has an $\mathfrak{I}_\Theta$-algebra structure. The following lemma gives some results on the k_0, $\mathfrak{K}$ and $\mathfrak{I}_\Theta$ dimensions of these algebras or vector spaces.

Lemma 1. For a $\mathfrak{K}$-vector space, χ_Θ is finite-dimensional. For k_0-algebras, L_Θ and $\mathcal{X}_\Theta$ are infinite-dimensional. For an $\mathfrak{I}_\Theta$-algebra, $\mathcal{X}_\Theta$ is finite-dimensional if and only if $\mathfrak{K}$ is algebraic over $\mathfrak{I}_\Theta$.

Proof. If the $\mathfrak{K}$-linearly independent elements $\theta_1, \theta_2, \ldots, \theta_r$ generate Θ then $X \in \mathcal{X}_\Theta$ satisfies $X \lfloor \theta_i = 0$ for $i = 1, \ldots, r$. Since T is finite of dimension $n + 1$ over $\mathfrak{K}$, $\mathcal{X}_\Theta$ has dimension $N - r$. $\mathfrak{K}$ is not finite-dimensional over k_0, hence $\dim_{k_0} L_\Theta \geqslant \dim_{k_0} \mathcal{X}_\Theta = \infty$. Finally,

$$\dim_{\mathfrak{I}_\Theta} \mathcal{X}_\Theta = \dim_{\mathfrak{K}} \mathcal{X}_\Theta \dim_{\mathfrak{I}_\Theta} \mathfrak{K}$$

and the first factor on the right is finite. □

We remark that the functions $\mathfrak{I}_\Theta$ are the rational invariants in $y_0, y_1, \ldots, y_{n-1}$ of the differential system over k. Also L_Θ consists of more than just *point* symmetries, that is, symmetries induced by transformations of the dependent and independent variables in a differential equation. Characteristic symmetries, for example, are not point symmetries. An example of a nonpoint, noncharacteristic symmetry is $y_1 \partial_{y_1}$ for the ideal generated by $\mathrm{d}y_0 - y_1\,\mathrm{d}x$ and $\mathrm{d}y_1 - (y_1)^{-1}\,\mathrm{d}x$.

All that has been said so far pertains to the field $\mathfrak{K}$, but it equally applies to extensions and when we wish to emphasize this we will indicate the extension appropriately. Thus, for example,

$$L_\Theta(\mathfrak{K}_1) = \{X \in \mathfrak{K}_1 T | \mathcal{L}_X \Theta \subseteq \mathfrak{K}_1 \Theta\}.$$

From the definition of the Lie derivative,

$$\mathcal{L}_X \omega = X \lfloor \mathrm{d}\omega + \mathrm{d}(X \lfloor \omega) \tag{4}$$

it is apparent that $\mathcal{X}_\Theta$ is a k_0-subalgebra of L_Θ. Even better, we have the following.

Lemma 2. $\mathcal{X}_\Theta \trianglelefteq L_\Theta$.

Proof. Let $X \in \mathcal{X}_\Theta$ and $Y \in L_\Theta$ then for any set of generating elements $\theta_i \in \Theta$,

$$\begin{aligned}[X, Y] \lfloor \theta_i &= \mathrm{d}\theta_i(X, Y) + Y(X \lfloor \theta_i) - X(Y \lfloor \theta_i) \\ &= Y \lfloor (X \lfloor \mathrm{d}\theta_i) - X(Y \lfloor \theta_i).\end{aligned}$$

But from the symmetry condition

$$Y \lfloor \mathrm{d}\theta_i + \mathrm{d}\,(Y \lfloor \theta_i) \in \Theta$$

we obtain $X \lfloor (Y \lfloor \mathrm{d}\theta_i) + X(Y \lfloor \theta_i) = 0$. □

Recall that the *idealizer*, $\mathbb{I}(\mathfrak{n})$, of a subalgebra $\mathfrak{n}$ of an algebra $\mathfrak{g}$ is the largest subalgebra of $\mathfrak{g}$ containing $\mathfrak{n}$ in which $\mathfrak{n}$ is an ideal. We omit the proof of the following theorem which gives an alternative definition of the Lie symmetry algebra.

Theorem 1. $L_\Theta = \mathbb{I}(\mathcal{X}_\Theta)$.

Definition 3. The *noncharacteristic Lie symmetry algebra* of Θ is $S_\Theta = L_\Theta/\mathcal{X}_\Theta$.

The Lie algebra S_Θ has no well defined action on Θ itself: if $X_1 - X_2 \in \mathcal{X}_\Theta$ it does not follow that $\mathcal{L}_{X_1}\theta = \mathcal{L}_{X_2}\theta$ in general. However, we do have the following generalization to S_Θ of standard theorems[10, 11].

Theorem 2. Assume S_Θ is finite-dimensional as an $\mathfrak{I}_\Theta$ vector space. If $\mathfrak{n}$ is a nontrivial subalgebra of S_Θ then there exists an ideal $\Theta_\mathfrak{n} \lhd \Theta$ with noncharacteristic symmetry algebra containing a subalgebra isomorphic to $\mathbb{I}(\mathfrak{n})/\mathfrak{n}$.

Proof. Associated with $\mathfrak{n}$ is a subalgebra $\mathcal{N}$ of L_Θ spanned, over $\mathfrak{I}_\Theta$ by noncharacteristic vector fields $X_1, X_2, \ldots, X_q$. Let $\Theta_\mathfrak{n}$ be the algebra ideal of $\bigwedge$ for which $\mathcal{N}$ is characteristic. So $\Theta_\mathfrak{n} \leqslant \Theta$, because $\mathcal{X}_\Theta \leqslant \mathcal{N}$. It is generated by one-forms $\theta_{q+1}, \theta_{q+2}, \ldots, \theta_r$, with $X_i \lfloor \theta_j = 0$ for $i = 1, \ldots, q$ and $j = q+1, \ldots, r$. We wish to show that $\Theta_\mathfrak{n}$ is a differential ideal and that $\mathcal{L}_X \Theta_\mathfrak{n} \subseteq \Theta_\mathfrak{n}$ for $X \in \mathcal{N}$.

Extending the basis of $\Theta_\mathfrak{n}$ to a $\mathfrak{K}$-basis of Θ we have:

$$\mathrm{d}\theta_i = \sum_{j=1}^{q} \alpha_i^j \wedge \theta_j + \sum_{j=q+1}^{r} \beta_i^j \wedge \theta_j$$

and

$$\mathcal{L}_{X_i}\theta_j = \sum_{k=1}^{q} \lambda_{ij}^k \theta_k + \sum_{k=q+1}^{p} \mu_{ij}^k \theta_k$$

where the coefficients in these relations all belong to $\bigwedge$. For $i, l = 1, \ldots, q$ and $j = q+1, \ldots, r$ the Lie derivative condition gives us

$$X_l \lfloor (X_i \lfloor \mathrm{d}\theta_j) = \sum_{k=1}^{q} \lambda_{ij} X_l \lfloor \theta_k.$$

The left-hand side vanishes, by the definition of the d operator on one-forms and the fact that $\mathcal{N}$ is a subalgebra of L_Θ. Since the matrix with entries $X_l \lfloor \theta_k$, $l, k = 1.\ldots, q$ is of rank q, we have $\lambda_{ij}^k = 0$ for $k = 1, \ldots, q$. Hence, the symmetries of Θ in $\mathcal{N}$ actually leave $\Theta_\mathfrak{n}$ (setwise) invariant.

Since then $X_i \lfloor \mathrm{d}\theta_j \in \Theta_\mathfrak{n}$, the closure condition on Θ gives

$$\sum_{k=1}^{q} ((X_i \lfloor \alpha_j^k)\theta_k - (X_i \lfloor \theta_k)\alpha_j^k) \in \Theta_\mathfrak{n}. \tag{5}$$

For convenience, choose the bases such that $X_i \lfloor \theta_k = \delta_{ik}$. From the above we then obtain

$$\alpha_j^i = \sum_{k=1}^{q} (X_i \lfloor \alpha_j^k)\theta_k \qquad \mathrm{mod}\ \Theta_\mathfrak{n}$$

so that we may write, for $j = q+1, \ldots, p$,

$$\mathrm{d}\theta_j = \sum_{l,k=1}^{q} (X_l \lfloor \alpha_j^k)\theta_k \wedge \theta_l \qquad \mathrm{mod}\ \Theta_\mathfrak{n}.$$

But from (5) the coefficients $X_l \lfloor \alpha_j^k$ are symmetric in l and k. Consequently $\mathrm{d}\Theta_{\mathfrak{n}} \subseteq \Theta_{\mathfrak{n}}$.

Finally, the noncharacteristic symmetry algebra of $\Theta_{\mathfrak{n}}$ is $S_{\mathfrak{n}} = \mathbb{I}(\mathcal{N})/\mathcal{N}$, the idealizer being relative to T. Because $\mathbb{I}(\mathcal{N}) \cap \mathbb{I}(\mathcal{X})$ is a subalgebra of $\mathbb{I}(\mathcal{N})$ containing $\mathcal{N}$, $S_{\mathfrak{n}}$ has subalgebra $(\mathbb{I}(\mathcal{N}) \cap \mathbb{I}(\mathcal{X}))/\mathcal{N}$ which is isomorphic to $\mathbb{I}(\mathfrak{n})/\mathfrak{n}$, the idealizer here being relative to S_Θ. □

Thus, for a symmetry subalgebra there is a reduced system. In general there is no simple relationship between the noncharacteristic Lie symmetry algebras of ideals $\Theta_1 \triangleleft \Theta$. Although $\mathcal{X}_\Theta < \mathcal{X}_{\Theta_1}$, a Lie derivative which preserves Θ need not preserve Θ_1 or vice versa.

We will need the following in a later section.

Lemma 3. An ideal, Θ, of rank one with a noncharacteristic symmetry is generated by a closed one-form.

Proof. Let s be the symmetry of Θ in S_Θ and let Θ be generated by θ over $\mathfrak{K}$. We have

$$\mathrm{d}\theta = \Gamma \wedge \theta$$

for Γ a $\mathfrak{K}$-valued one-form. Consider the exterior derivative of $(s \lfloor \theta)^{-1}\theta$:

$$\begin{aligned}(s\lfloor\theta)^2 \mathrm{d}\left(\frac{\theta}{s\lfloor\theta}\right) &= (s\lfloor\theta)\,\mathrm{d}\theta - \mathrm{d}\,(s\lfloor\theta) \wedge \theta \\ &= (s\lfloor\theta)\,\mathrm{d}\theta + (s\lfloor \mathrm{d}\theta) \wedge \theta \\ &= s\lfloor(\theta \wedge \mathrm{d}\theta)\end{aligned}$$

which vanishes by the closure of Θ. Clearly $s\lfloor\theta \in \mathfrak{K}$ and the result follows. □

Locally and away from singularities we can find an integral for this closed form but it will generally dwell in an extension of $\mathfrak{K}$.

Theorem 3. If the single vector field $s \in S_\Theta$ generates a subalgebra $\mathfrak{n}$ then, in the context of theorem 2, $\Theta/\Theta_{\mathfrak{n}}$ has a closed generator.

Proof. This is the case $q = 1$ in theorem 2, so $\Theta/\Theta_{\mathfrak{n}}$ has rank one as a $\mathfrak{K}$-algebra, generated by some $\theta + \Theta_{\mathfrak{n}}$ with $\mathrm{d}\theta = \Gamma \wedge \theta \bmod \Theta_{\mathfrak{n}}$ by theorem 2. Now apply lemma 3.

3. The general linear equation of order n

The theory of the previous section applies to quite general differential equations. We now specialize to linear equations of the form

$$z^{(n)} + a_{n-1}(x)z^{(n-1)} + a_{n-2}(x)z^{(n-2)} + \cdots + a_1 z^{(1)} + a_0 z = 0 \tag{6}$$

where $z^{(j)}$ is the jth differential coefficient of z and $a_i(x)$ belong to k. The associated exterior differential ideal is generated over $\mathfrak{K}$ by the n one-forms $\theta_1 = \mathrm{d}y_0 - y_1\,\mathrm{d}x, \theta_2 = \mathrm{d}y_1 - y_2\,\mathrm{d}x, \ldots, \theta_{n-1} = \mathrm{d}y_{n-2} - y_{n-1}\,\mathrm{d}x, \theta_n = \mathrm{d}y_{n-1} + \sum_{i=0}^{n-1} a_i y_i\,\mathrm{d}x$.

Let $z_1, z_2, \ldots, z_n$ be a set of solutions to (6) which are linearly independent over k_0 and let k_1 be a differential field extension of k large enough to contain them, say the Picard–Vessiot extension [4]. Define associated vector fields in $\mathfrak{K}_1 T$,

$$X_i = z_i\,\partial_{y_0} + z_i^{(1)}\partial_{y_1} + \cdots + z_i^{(n-1)}\partial_{y_{n-1}} \qquad i = 1, \ldots, n. \tag{7}$$

Note that $X_j \lfloor \theta_i$ are not zero in k_1 and so X_i cannot be characteristic.

We replace $\mathfrak{K}$ by $\mathfrak{K}_1$ in all the definitions and results of the previous section. The following lemma and theorem are fundamental.

Lemma 4. The z_i are linearly independent over $\mathfrak{I}_\Theta$.

Proof. (This actually holds over $\mathfrak{K}_1$ as well.) Let $\sum_{i=1}^n \alpha_i X_i = 0$. Then because $X_j \lfloor \theta_i = z_j^{(i-1)}$, for $i, j = 1, \ldots, n$ we have the n equations

$$\sum_{i=1}^{n} \alpha_i z_j^{(i-1)} = 0 \qquad j = 1, \ldots, n$$

which have only a trivial solution by the nonvanishing of the Wronskian. □

Theorem 4. S_Θ is generated as an $\mathfrak{I}_\Theta$-vector space by $X_1, X_2, \ldots, X_n$.

Proof. First, we show that $X = \sum_{i=1}^n \alpha_i X_i \in S_\Theta$ for $\alpha_i \in \mathfrak{I}_\Theta$. Thus, for $k = 1, \ldots, n-1$, $\mathcal{L}_{X_i}\theta_k = -z_i^{(k)}\,\mathrm{d}x + \mathrm{d}z_i^{(k-1)} = 0$, whilst for θ_n,

$$\mathcal{L}_{X_i}\theta_n = -\sum_{j=0}^{n-1} a_j z_i^{(j)}\,\mathrm{d}x - z_i^{(n)}\,\mathrm{d}x = 0.$$

Further, for $f \in \mathfrak{K}_1$, $\mathcal{L}_{fX}\theta = f\mathcal{L}_X\theta + (X\lfloor\theta)\,\mathrm{d}f \in \Theta$ provided $f \in \mathfrak{I}_\Theta$. In addition, if X were characteristic we would have $\sum_{i=1}^n \alpha_i z_i = 0$ which contradicts the linear independence of the z_i by lemma 4.

Conversely, suppose $S = Z\partial_x + Y_0\partial_{y_0} + \cdots + Y_{n-1}\partial_{n-1} \in L_\Theta$ is noncharacteristic. Then $n-1$ of the symmetry conditions on S are

$$\mathrm{d}\,(Y_i - Zy_{i+1}) + Z\,\mathrm{d}y_{i+1} - Y_{i+1}\,\mathrm{d}x \in \Theta \qquad i = 0, \ldots, n-2. \tag{8}$$

Using instead, and without loss of generality, the homomorphic image of X in S_Θ (so we take $Z = 0$) we obtain

$$\mathrm{d}Y_i - Y_{i+1}\,\mathrm{d}x \in \Theta \qquad i = 0, \ldots, n-2.$$

The remaining symmetry condition is

$$\mathrm{d}Y_{n-1} - \sum_{i=0}^{n-1} a_i Y_i\,\mathrm{d}x \in \Theta.$$

We now show that these equations are a linear system of rank n over the field of invariants. We do this by replacing the y_i dependence of the Y_i by a dependence on certain chosen invariants. For $i = 1, \ldots, n$ define

$$I_i = \begin{vmatrix} z_1 & \cdots & z_{i-1} & y_0 & z_{i+1} & \cdots & z_n \\ z_1^{(1)} & \cdots & z_{i-1}^{(1)} & y_1 & z_{i+1}^{(1)} & \cdots & z_n^{(1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1^{(n-1)} & \cdots & z_{i-1}^{(n-1)} & y_{n-1} & z_{i+1}^{(n-1)} & \cdots & z_n^{(n-1)} \end{vmatrix}.$$

It is straightforward to check that $X(I_i) = 0$ for $X \in \mathcal{X}_\Theta$ and we may solve for the y_i in terms of the I_i over k_1. Hence, we write $Y_i = Y_i(x, I_1, \ldots, I_n)$ with $\mathrm{d}I_i \in \Theta, \forall i$. As functions of x alone, we obtain

$$Y_{i,x} - Y_{i+1} = 0 \qquad i = 0, \ldots, n-2$$
$$Y_{n-1,x} - \sum_{i=0}^{n-1} a_i Y_i = 0$$

which are just the equations of the linear differential equation we started with, written in system form. □

The fact that the X_i are Lie symmetries of the nth-order linear equation is known in the literature [12, 13]. The significance of the above result is threefold. First, because the X_i are all point symmetries, it states that, for the linear equations, *every* (generalized) Lie symmetry is generated over the invariants by point symmetries modulo the characteristic symmetries. Secondly, it gives a clear picture of the structure of S_Θ for all values of n. Thirdly, it will allow us to exhibit the crucial role played by the intermediate extensions, $\mathfrak{K}_2$ etc, $\mathfrak{K} \to \mathfrak{K}_2 \to \mathfrak{K}_1$, to which subsets of invariants, or first integrals, of the system, belong.

Note that S_Θ is finite-dimensional as an $\mathfrak{I}_\Theta$-vector space and infinite-dimensional as a k_0-algebra. We emphasize this because ordinary differential equations have finite-dimensional point symmetry algebras.

Note also that whereas L_Θ acts on Θ, S_Θ has no well-defined action on the whole of Θ. But one easily sees that $X_i \lfloor \mathrm{d}I_j = \delta_{ij}$ and that the X_i commute. In fact then S_Θ acts in a well-defined manner on $\tilde{\bigwedge} = \mathfrak{I}_\Theta \langle \mathrm{d}I_1, \ldots, \mathrm{d}I_n \rangle$, the subalgebra of invariant forms inside Θ. $\tilde{\bigwedge}$ is the algebra of rational forms on an n-dimensional manifold for which S_Θ is the ring of derivations or tangent space. Of course, this is an object on which we would really like to obtain a grip: the solution manifold. The I_i are also the constants of integration or first integrals familiar from elementary existence theory.

4. A result from differential algebra

When dealing with second-order equations in this section we will use x, y and p for x, y_0 and y_1 and primes for derivatives with respect to x.

Consider, for example, the simplest nontrivial case, the second-order equation $z'' + az = 0$ for $a \in k$. It can be verified [13], with a little labour, that the Lie point symmetry algebra (X of the form $\eta(x, y)\partial_x + \zeta(x, y)\partial_y + (\zeta_{,x} + (\zeta_{,y} - \eta_{,x})p - \eta_{,y}p^2)\partial_p$) over k_0, is generated by the eight elements of $\mathfrak{K}_1 T$:

$$A_i = f_i\partial_x + \tfrac{1}{2}yf_i'\partial_y + \tfrac{1}{2}(f_i''y - f_i'p)\partial_p \qquad i = 1, 2, 3$$
$$B_i = yz_i\partial_x + y^2z_i'\partial_y + (pyz_i' - (p^2 + ay^2)z_i)\partial_p \qquad i = 1, 2$$
$$X_i = z_i\partial_y + z_i'\partial_p \qquad i = 1, 2$$
$$D = y\partial_y + p\partial_p$$

where f_1, f_2 and f_3 are k_0-linearly independent solutions to the third-order equation,

$$f''' + 4af' + 2a'f = 0 \tag{9}$$

and z_1 and z_2 are k_0-linearly independent solutions of the original second-order equation,

$$z'' + az = 0. \tag{10}$$

In fact, the general solution of (9) is a k_0 quadratic form in the solutions of (10), so we may take $f_1 = z_1^2$, $f_2 = z_1z_2$ and $f_3 = z_2^2$.

Up to multiplication by an arbitrary element of $\mathfrak{K}_1$ there is one element in $\mathcal{X}_\Theta$ which we take to be

$$X = \partial_x + p\partial_y - ay\partial_p$$

and it is then easy to show that the elements of the Lie point symmetry algebra decompose, over $\mathfrak{I}_\Theta$, in terms of X, X_1 and X_2. For example,

$$D = (yz_1' - pz_1)X_2 - (yz_2' - pz_2)X_1$$
$$A_1 = z_1^2X + (yz_1' - z_1p)X_1$$
$$B_1 = yz_1X - (yz_1' - z_1p)^2X_2 - (yz_1' - z_1p)(yz_2' - z_2p)X_1$$

etc.

By the result of theorem 4 we know that we do not need to worry about any other possible (nonpoint) symmetries of Θ. The full noncharacteristic symmetry algebra of this equation is therefore all vector fields of the form,

$$F_1(I_1, I_2)(z_1\partial_y + z_1'\partial_p) + F_2(I_1, I_2)(z_2\partial_y + z_2'\partial_p).$$

This is a large noncommutative k_0-algebra.

We will now study, for this example, noncharacteristic Lie symmetries with coefficients in intermediate extensions $\mathfrak{K}_2$. Consider the following tower of fields:

$$\begin{array}{ccc} k_1 & = & k\langle z_1, z_2\rangle \\ | & & \\ k_2 & = & k\langle f_1, f_2, f_3\rangle \\ | & & \\ k & & \end{array}$$

where z_1 and z_2 are linearly independent (over k_0) solutions to (10), f_1, f_2 and f_3 linearly independent (over k_0) solutions to (9) and the angle brackets denote the appropriate Picard–Vessiot extensions. There is a corresponding tower of fields of invariants:

$$\begin{array}{c} \mathfrak{I}_\Theta(\mathfrak{K}_1) \\ | \\ \mathfrak{I}_\Theta(\mathfrak{K}_2) \\ | \\ \mathfrak{I}_\Theta(\mathfrak{K}). \end{array}$$

Lemma 5. $[\mathfrak{I}_\Theta(\mathfrak{K}_1) : \mathfrak{I}_\Theta(\mathfrak{K}_2)] = 2$.

Proof. Writing y and p as functions of I_1 and I_2 over k_1, as in the proof of theorem 2, and using their invariance, we see that $\mathfrak{I}_\Theta(\mathfrak{K}_1) = k_0(I_1, I_2)$. The $\mathfrak{K}_2$ invariants are a subfield of this. But I_1^2, $I_1 I_2$ and I_2^2 all belong to $\mathfrak{I}_\Theta(\mathfrak{K}_2)$. For instance,

$$I_2^2 = y^2 z_1'^2 - yp(z_1^2)' + p^2 z_1^2$$

and since

$$2z_1'^2 = (z_1^2)'' - 2z_1 z_1'' = (z_1^2)'' + 2az_1^2$$

I_2^2 belongs to the differential field $\mathfrak{K}\langle z_1^2, z_1 z_2, z_2^2\rangle$. Because the quadratic expressions in the z's are linearly independent and solve the third-order equation, this is just $\mathfrak{K}\langle f_1, f_2, f_3\rangle = \mathfrak{K}_2$. □

The symmetry algebra associated with $\mathfrak{K}_2$, $S_\Theta(\mathfrak{K}_2)$ is spanned over $\mathfrak{I}_\Theta(\mathfrak{K}_2) = k_0(I_1^2, I_1 I_2, I_2^2)$ by D and A_1, A_2 and A_3.

The symmetry algebra $S_\Theta(\mathfrak{K})$ contains at least the generator

$$D = y\partial_y + p\partial_p = -I_2 X_2 - I_1 X_1.$$

We apply theorem 2 to construct a closed ideal $\Theta' \triangleleft \Theta$ using D. Thus, we may take $\theta_1 = \mathrm{d}y - p\,\mathrm{d}x$ and $\theta_2 = p\,\mathrm{d}y - y\,\mathrm{d}p - (p^2 + ay^2)\,\mathrm{d}x$. D is characteristic for $\Theta' = \mathfrak{K}\theta_2$ and

$$\mathrm{d}\theta_2 = \frac{2}{y}\theta_1 \wedge \theta_2.$$

By theorem 2, Θ' has as a noncharacteristic symmetry any element of the idealizer in $S_\Theta(\mathfrak{K})$ of the subalgebra spanned by D. The general representation of the elements of $S_\theta(\mathfrak{K}_1)$ given in theorem 4 is useful to identify this idealizer.

To do this it is sufficient to solve the equation

$$[\alpha X_1 + \beta X_2, D] = \gamma D$$

for α, β and γ functions in $k_0(I_1, I_2)$. Up to the characteristic subspace we may take the solution to be $\delta(I_1^2 + I_2^2)(-I_2 X_1 + I_1 X_2)$ which actually belongs to $S_\Theta(\mathfrak{K}_2)$. Thus, if the third-order equation (9) has a solution, $f \in K$, we may use it to find a closed one-form generating Θ' as in lemma 3. An alternative form of this criterion is to be found in lemma 6 below.

Applying the proof of lemma 3, the integrating factor is $((-I_2 X_1 + I_1 X_2)\lfloor\theta_2)^{-1} = (I_1^2 + I_2^2)^{-1}$ and hence

$$\frac{p\,\mathrm{d}y - y\,\mathrm{d}p - (p^2 + ay^2)\,\mathrm{d}x}{I_1^2 + I_2^2}$$

is closed. Away from singular points we may call it $\mathrm{d}F$ where F lives in a large enough extension of $\mathfrak{K}$. In fact,

$$F = k + \ln\left(\frac{I_1}{I_2}\right) = k + \ln\left(\frac{2pf - yf' + y}{2pf - yf' - y}\right) + \ln\left(\frac{z_1}{z_2}\right)$$

for $k \in k_0$ where we have used the general form $f = z_1 z_2$ to derive this result and have expressed F in terms of f, f' and the ratio z_1/z_2.

The one-form θ_2 is then anulled on the manifolds,

$$\frac{z_1}{z_2}\frac{2pf - yf' + y}{2pf - yf' - y} \in k_0$$

and on any such manifold,

$$\theta_1 = \mathrm{d}y - \frac{1}{2}\left(\frac{f'}{f} + \frac{z_2 + z_1}{f(z_2 - z_1)}\right)\mathrm{d}x.$$

A trivial application of lemma 2, using the symmetry D, then tells us to look at the closed one-form θ_1/y which states that y belongs to a Liouville extension of a field containing the ratio z_1/z_2. In the case we are considering, $f \in k$, it is easy to see that $k\langle\frac{z_1}{z_2}\rangle$ is itself a Liouville extension of k because

$$\left(\frac{z_1}{z_2}\right)^{(1)} = \frac{1}{f}\frac{z_1}{z_2}.$$

Hence we have, purely on the basis of the noncharacteristic Lie theory, the following result.

Theorem 5. If the third-order equation $f''' + 4af' + 2a'f = 0$ has only one solution in k, up to a factor in k_0, then the second-order equation $z'' + az = 0$ defines a generalized Liouville extension of k.

Note that the condition that only one solution of the third-order equation be in k is necessary; otherwise the extension $k\langle z_1, z_2\rangle$ is of finite degree over k and algebraic rather than Liouville.

This result is equivalent to theorem 6.4 of [4] by the following lemma.

Lemma 6. The Ricatti equation $\phi' = \phi^2 + a$ has a solution in a quadratic extension of k if and only if the third-order equation $f''' + 4af' + 2a'f = 0$ has a solution in k.

Proof. Suppose the third-order equation has a solution in k. Call it $f = z_1 z_2$ and define $t_1 = z_1'/z_1$, $t_2 = z_2'/z_2$. Then $t_1 + t_2 = (z_1 z_2)'/(z_1 z_2) \in k$. Also, we may obtain expressions for z_i', z_i'' and z_i'' in terms of the t_i, a and its derivatives:

$$\begin{aligned} z_i' &= t_i z_i \\ z_i'' &= (2t_i^2 + a)z_i \\ z_i''' &= (6t_i^3 + 5at_i + a')z_1. \end{aligned}$$

The third-order equation then gives the algebraic relation,

$$6(t_1 + t_2)^3 - 12t_1 t_2 (t_1 + t_2) + 12a(t_1 + t_2) + 2a' = 0$$

from which $t_1 t_2 \in k$. Thus, the Ricatti equation has (a pair of) solutions in a quadratic extension of k.

Suppose, conversely, that $t' = t^2 + a$ has a solution in such a quadratic extension, so that

$$t^2 + rt + s = 0$$

for r and s in k. From $t_1 + t_2 = -r$ and $t_1 t_2 = s$ we obtain

$$t_1 z_1 + t_2 z_2 = -r' \qquad z_1 + z_2 = s'/s.$$

Solving for z_1 and z_2 we find

$$f = z_1 z_2 = \frac{sr'^2 + s'^2 + rr's'}{s(r^2 - 4s)}$$

which is in k. □

Finally, we note that in the above example the noncharacteristic Lie symmetry algebra of the second-order equation contains a subalgebra with coefficients in k which is solvable. It is this feature which allows us to conclude the integrability of the equation in a generalized Liouville extension.

5. Conclusion

We have defined and used the noncharacteristic Lie symmetry algebra for a linear differential equation to study the algebraic structure of the field extensions defined by its solutions.

Historically, Lie's theory for differential equations was a twin to the differential Galois theory of Vessiot and Picard, later developed by Ritt [15] and Kolchin [14]. Modern treatments are to be found in Kaplansky [4] and Magid [8]. Differential Galois theory, in contrast to Lie's, was mainly developed for homogeneous, linear equations in a single dependent variable and unlike the Lie theory, close attention is paid to the differential field extensions to which the solutions belong. The coefficients of the equation are taken to belong to a differential ground field. The differential Galois group is the group of differential automorphisms of an extension, containing all the solutions, fixing the ground field. As such, this group is an algebraic subgroup of a general linear group. The Lie symmetry group of the general linear equation does not look like this; in particular it does not fix the ground field of coefficients. Nevertheless, similar results concerning 'solvablility' hold for the differential Galois group.

In this paper we have looked at the Lie symmetry theory from an algebraic point of view and have shown, by example, that this can lead to concrete results. However, the machinery involved is (superficially, at least) to be distinguished from that of differential Galois theory because the symmetry groups themselves are quite distinct in the two cases.

There are two clear paths for further study: first, to develop the connection between S_Θ and the differential Galois group; secondly, to see if S_Θ for classes of nonlinear equation has as natural and as useful a structure as is the case for linear equations.

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